Homework 4

#1 From an earlier homework, there exists a space X such that $\pi(X, x_0) = G$. (By the way, this is true for *any* group G.) Let $\tilde{X}, \tilde{x}_0$ be the based covering space corresponding to $H = N \subseteq \pi(X, x_0)$. Then $\tilde{X}$ is normal and $\mathcal{A}(\tilde{X}) \approx G/N$.

#3 Let $H \subseteq \pi(X, x_0)$ be the commutator subgroup (which is normal) and $\tilde{X}_a$ the normal covering corresponding to H. Then $\mathcal{A}(\tilde{X}_a) \approx \pi(X, x_0)/H$ is abelian. If $\tilde{X}$ is a covering such that $\mathcal{A}(\tilde{X})$ is abelian, then $\pi(X, x_0)/p_* \pi(\tilde{X}, \tilde{x})$ is abelian. Therefore $p_{a*}\pi(\tilde{X}_a, \tilde{x}_a) \subseteq p_* \pi(\tilde{X}, \tilde{x})$ $\therefore$ there exists a homomorphism $\varphi: \tilde{X}_a \to \tilde{X}$ such that $p\varphi = p_a$.

#4 The subgroups of $\mathbb{Z}$ are $n\mathbb{Z}$, $n \geq 0$. The subgroups of $\mathbb{Z}_2$ are $\{1\}$ and $\mathbb{Z}_2$. We know the covering spaces for each of these subgroups. For the third example, the total space X is a closed annulus containing a circle as dr. $\therefore$ Subgroups of the fundamental group are $n\mathbb{Z}$, $n \geq 0$. The covering spaces (other than X) are helicoids. These can be visualized by taking the midpoint of a ~~____~~ closed line segment L of length 1 and having it trace out a surface as the center of L moves around a helix. If the subgroup is $n\mathbb{Z}$, $n \geq 1$, go around the helix n times and identify the top and bottom (much as was done ~~__~~ for the covering p_n of S^1).

#5 Define $\theta: K \to \mathcal{A}(\tilde{G})$ by $\theta(k) = \varphi_k$.

1. φ_k is a homo. : $p\varphi_k(x) = p(x \cdot k) = px \cdot pk = px \cdot e = p(x)$
2. θ is a homo. : $k, l \in K$

   $(\varphi_l \varphi_k)(x) = \varphi_l(kx) = (lk) \cdot x = \varphi_{lk}(x)$ so $\theta(l)\theta(k) = \theta(lk)$
3. θ is onto: Let $\varphi \in \mathcal{A}(\tilde{G})$ let $k = \varphi(e)$ $\varphi_k(e) = ek = \varphi(e)$

   $\therefore \varphi = \varphi_k$ so $\varphi = \theta(k)$.

4. θ is one-one: Suppose $\theta(k) = \theta(l)$ $\therefore \varphi_k = \varphi_l$. Apply to e.

#6 (a) This is a 2-sheeted cover. $\therefore$ the index of $p_* \pi(E)$ in $\pi(X)$ is 2.
But any subgroup of index 2 is normal.

(b) Read from left to right and label the vertices e_{-1}, e_0, e_1 and the circles c_1, c_2, c_3, c_4. If $h \in \mathcal{A}(E)$ and $h \neq id$, then $h(e_{-1}) = e_0$ or e_1. But at e_{-1} there is one loop which projects onto B ~~and one loop which~~ Neither e_0 nor e_1 has this property.
$\therefore h = id$ so $\mathcal{A}(E)$ is trivial.

#7 Fix $x_0 \in X$ and define $\theta: G \to X$ by $\theta(g) = x_0 g$. By transitivity, θ is onto. Show $\theta(G_{x_0} g) = \theta(g)$ $\therefore$ θ induces $\theta': G/G_{x_0} \to X$ which is onto. Show θ' is one-one.